

# PHY306: Homework#11 Solutions, Spring 10

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## 1 Problem 7.3(10pts)

Neglecting both spin (which cancels out of the final result) and the excited states of the hydrogen atom (which contribute negligibly even at 10,000K), this system has just two states:

Unoccupied:  $E=0$ ,  $N=0$ , Gibbs Factor= $e^0=1$ ;

occupied:  $E=-I$ ,  $N=1$ , Gibbs factor= $e^{-(-I-\mu)/kT}$ .

The ratio of the probabilities of these two states is the same as the ratio of partial pressure of ionized to un-ionized atoms:

$$\frac{P_p}{P_H} = \frac{P(\text{unoccupied})}{P(\text{occupied})} = \frac{1}{e^{-(-I-\mu)/kT}} = \frac{e^{-I/kT}}{e^{\mu/kT}}.$$

Here  $\mu$  is the chemical potential for electrons. Treating the electron as an ideal gas and again neglecting spin,

$$\mu = -kT \ln \left( \frac{V Z_{int}}{N v_Q} \right) = -kT \ln \left( \frac{kT}{P_e v_Q} \right),$$

so  $e^{\mu/kT} = P_e v_Q / kT$ , which yields the Saha Equation,

$$\frac{P_p}{P_H} = \frac{kT}{P_e v_Q} e^{-I/kT}.$$

## 2 Problem 7.4(5pts)

Taking electron spin in to account, the hydrogen atom now has two occupied states, each with the same energy, so the ratio of unoccupied to occupied atoms is

$$\frac{P_p}{P_H} = \frac{P(\text{unoccupied})}{P(\text{occupied})} = \frac{1}{2e^{-(I-\mu)/kT}} = \frac{e^{-I/kT}}{2e^{\mu/kT}}.$$

But now, a free electron has two degenerate states, so the chemical potential of the electron gas is

$$\mu = -kT \ln \left( \frac{V Z_{int}}{N v_Q} \right) = -kT \ln \left( \frac{2kT}{P_e v_Q} \right),$$

implying that  $e^{\mu/kT} = P_e v_Q / 2kT$ . When we plug this into equation (1), the factors of 2 cancel, yielding the Saha equation exactly as derived in the previous problem.

## 3 Problem 7.9(10pts)

The mass of an  $N_2$  molecule is about 28 atomic mass units, so its quantum volume at room temperature is

$$\begin{aligned} v_Q &= \left( \frac{h^2}{2\pi m k T} \right)^{3/2} = \left( \frac{(6.63 \times 10^{-34} \text{ J} \cdot \text{s})^2}{2\pi (1.66 \times 10^{-27} \text{ kg})(1.38 \times 10^{-23} \text{ J/K})(300 \text{ K})} \right)^{3/2} \\ &= 6.9 \times 10^{-33} \text{ m}^3 = (1.9 \times 10^{-11} \text{ m})^3 \end{aligned}$$

(since atoms are about  $10^{-10}$  meters wide, this is substantially smaller than the physical volume of the nitrogen molecule.) Now we can use Boltzmann statistics whenever  $Z_1 \gg N$ , that is,  $V \gg N v_Q$ . But at standard temperature and pressure,

$$\frac{V}{N} = \frac{kT}{P} = \frac{(1.38 \times 10^{-23} \text{ J/K})(300 \text{ K})}{10^5 \text{ Pa}} = 4.1 \times 10^{-26} \text{ m}^3.$$

This is greater than the quantum volume by a factor of about 6 million, so Boltzmann statistics should be very accurate. On the other hand, Boltzmann statistics would break down when  $v_Q \approx V/N$ . Holding  $V/N$  fixed (i.e., fixed density), this would require that  $T^{3/2}$  be lower by a factor of 6 million or that  $T$  be lower by a factor of about 30,000. That's 1/100K. In other words, quantum statistics is irrelevant to an ordinary gas at this density provided that the temperature is higher than 0.01K.

## 4 Problem 7.10(20pts)

(A five-particle system with evenly spaced energy levels.)

(a) If the particles are distinguishable, then all five will settle into the lowest energy level. The same will happen if they are indistinguishable bosons. If they are indistinguishable fermions, however, then only one may occupy each level so each of the five lowest levels will contain one particle. The occupancies of the five lowest levels are therefore:

Distinguishable or Bosons		Fermions
5 0 0 0 0		1 1 1 1 1

(b) If the particles are distinguishable or bosons, then the system's first excited state has one of the five particles promoted to the second-lowest level. For distinguishable particles, there are *five* different ways to do this (choose any one of the five particles to promote), while for identical bosons there is only *one* way (it's meaningless to ask which particle has been promoted). For the system of fermions, the first excited state has the highest-energy particle promoted from the fifth state to the sixth (again adding just one unit of energy), and again there is only one way to do this. Graphically,

Distinguishable or Bosons		Fermions
4 1 0 0 0 0		1 1 1 1 0 1

(c) To add another unit of energy to the systems of bosons or distinguishable particles, we can either promote a second particle to the second-lowest level (leaving three in the lowest level) or leave four in the lowest level and promote the fifth up two levels. For bosons, there is only one way to do either of these things, but for distinguishable particles there are ten ways (5 choose 2) of doing the first and five of doing the second, for a total degeneracy of 15. Meanwhile, for the fermionic system there are two ways of putting in two units of energy, as illustrated below:

Distinguishable or Bosons		Fermions
3 2 0 0 0 0 0		1 1 1 0 1 1 0
4 0 1 0 0 0 0		1 1 1 1 0 0 1

To add yet another unit of energy there are basically three choices for each system, which I will simply illustrate in Table. 1 below. Again there is a large degeneracy for the system of distinguishable particles: 10 for the first arrangement (5 choose 3), 20 for the second (5 choices for the highest-energy particle and 4 for the next-highest), and 5 for the third for a total of 35.

(d) The probability for the system to be in any particular state is proportional to the Boltzmann factor for that state,  $e^{-E/kT}$  (where E is the total energy of all the particles). For any given E value, this quantity is the same for either system (and furthermore the allowed E values are the same). But the probability of finding the system with three units of energy is also proportional to the degeneracy, which is 3 for the bosonic system but 35 for the system of distinguishable particles. This means that at low temperatures we are much more likely to find a few units of energy in the system of distinguishable particles than in the system of bosons. Put it in another way, the ground state of the bosonic system is much more probable (relative to collection of all the low-energy excited states) than we might otherwise guess.

Distinguishable or Bosons	Fermions
2 3 0 0 0 0 0 0	1 1 0 1 1 1 0 0
3 1 1 0 0 0 0 0	1 1 1 0 1 0 1 0
4 0 0 1 0 0 0 0	1 1 1 1 0 0 0 1

Table 1: Table for 7.10(c)

## 5 Problem 7.11(5pts)

The probability of a state being occupied is given by the Fermi-Dirac distribution function,  $\frac{1}{e^{(\epsilon-\mu)/kT}+1}$ . At room temperature,  $kT=0.026\text{eV}$ , so the probabilities are:

- (a) For  $\epsilon-\mu=-1\text{eV}$ , probability =  $\frac{1}{e^{-1/0.026}+1} = (1 + 2 \times 10^{-17})^{-1} = 1 - 2 \times 10^{-17} \approx 1$ .
- (b) For  $\epsilon-\mu=-0.01\text{eV}$ , probability =  $\frac{1}{e^{-0.01/0.026}+1} = \frac{1}{1.68} = 0.59$ .
- (c) For  $\epsilon-\mu=0$ , probability =  $\frac{1}{e^0+1} = \frac{1}{2} = 0.50$ .
- (d) For  $\epsilon-\mu=+0.01\text{eV}$ , probability =  $\frac{1}{e^{+0.01/0.026}+1} = \frac{1}{2.47} = 0.41$ .
- (e) For  $\epsilon-\mu=+1\text{eV}$ , probability =  $\frac{1}{e^{+1/0.026}+1} = \frac{1}{5 \times 10^{16}} = 2 \times 10^{-17}$ .

## 6 Problem 7.13(20pts)

The average occupancy of a state is given by the Bose-Einstein distribution function,

$$\bar{n} = \frac{1}{e^x - 1} = \frac{e^{-x}}{1 - e^{-x}}, \text{ where } x = (\epsilon - \mu)/kT.$$

The probability of a state being occupied by exactly  $n$  particles is

$$P(n) = \frac{e^{-nx}}{Z} = (e^{-x})^n (1 - e^{-x}).$$

Thus we can compute everything we need from the quantity  $e^{-x} = e^{-(\epsilon-\mu)/kT}$ , where in this case  $kT=0.026\text{eV}$ .

- (a) For  $\epsilon-\mu=0.001\text{eV}$ ,  $x=0.001/0.026=0.038$  and  $e^{-x}=0.962$ , so

$$\begin{aligned}\bar{n} &= \frac{0.962}{1 - 0.962} = 25.5; P(0) = 1 - 0.962 = 0.038, P(1) = (0.962)(1 - 0.962) = 0.036, \\ P(2) &= (0.962)^2(1 - 0.962) = 0.035, P(3) = (0.962)^3(1 - 0.962) = 0.054.\end{aligned}$$

(b) For  $\epsilon - \mu = 0.01 \text{ eV}$ ,  $x = 0.01/0.026 = 0.38$  and  $e^{-x} = 0.681$ , so

$$\begin{aligned}\bar{n} &= \frac{0.681}{1 - 0.681} = 2.13; P(0) = 1 - 0.681 = 0.319, P(1) = (0.681)(1 - 0.681) = 0.217, \\ P(2) &= (0.681)^2(1 - 0.681) = 0.148, P(3) = (0.681)^3(1 - 0.681) = 0.101.\end{aligned}$$

(c) For  $\epsilon - \mu = 0.1 \text{ eV}$ ,  $x = 0.1/0.026 = 3.8$  and  $e^{-x} = 0.0214$ , so

$$\begin{aligned}\bar{n} &= \frac{0.0214}{1 - 0.0214} = 0.0218; P(0) = 1 - 0.0214 = 0.979, P(1) = (0.0214)(1 - 0.0214) = 0.021, \\ P(2) &= (0.0214)^2(1 - 0.0214) = 0.00045, P(3) = (0.0214)^3(1 - 0.0214) = 0.000010.\end{aligned}$$

(d) For  $\epsilon - \mu = 1 \text{ eV}$ ,  $x = 1/0.026 = 38$  and  $e^{-x} = 2 \times 10^{-17}$ , so

$$\begin{aligned}\bar{n} &= \frac{2 \times 10^{-17}}{1} = 2 \times 10^{-17}; P(0) = 1 - 2 \times 10^{-17} \approx 1, \\ P(1) &= (2 \times 10^{-17})(1 - 2 \times 10^{-17}) = 1 - 2 \times 10^{-17}, \\ P(2) &= (2 \times 10^{-17})^2(1 - 2 \times 10^{-17}) = 4 \times 10^{-34}, P(3) = (2 \times 10^{-17})^3(1 - 2 \times 10^{-17}) = 8 \times 10^{-51}.\end{aligned}$$