

# PHY306: Homework#2 Solutions, Spring 10

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## 1 Problem 1.62(5+5pts)

(a) Consider the segment of the rod in the Fig. 1, which is centered on point  $x$  with width  $\Delta x$ . The rate of heat flow in to this segment from the left is,

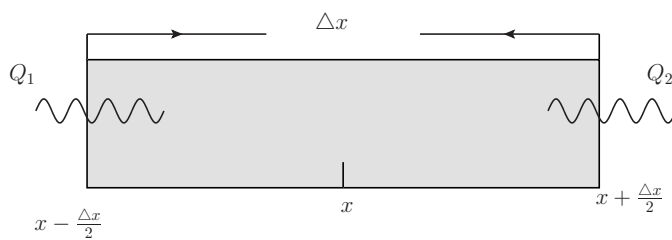


Figure 1: heat transfer in 1.62

$$\frac{Q_1}{\Delta t} = -k_t A \frac{dT}{dx} \Big|_{x - \frac{\Delta x}{2}}. \quad (1)$$

(according to the Fourier heat conduction law), where  $A$  is the cross-sectional area of the rod. Similarly, the rate of heat flow into the segment from the right is,

$$\frac{Q_2}{\Delta t} = +k_t A \frac{dT}{dx} \Big|_{x + \frac{\Delta x}{2}}, \quad (2)$$

with a plus sign because I am defining  $Q_2$  to be positive when heat flows *into* the segment. Thus the total rate of heat flow into the segment is

$$\frac{Q}{\Delta t} = k_t A \left( \frac{dT}{dx} \Big|_{x+\frac{\Delta x}{2}} - \frac{dT}{dx} \Big|_{x-\frac{\Delta x}{2}} \right) = k_t A \Delta x \frac{d^2 T}{dx^2}. \quad (3)$$

On the other hand, the heat input  $Q$  can be written as  $C\Delta T$ , where  $C$  is the heat capacity of the segment. The heat capacity, in turn, can be written as  $mc$ , where  $m$  is the mass of the segment and  $c$  is the specific heat. Therefore the previous equation becomes

$$\frac{mc\Delta T}{\Delta t} = k_t A \Delta x \frac{d^2 T}{dx^2}. \quad (4)$$

But  $m/(A\Delta x)$  is just the density of the rod,  $\rho$ , while  $\Delta T/\Delta t$  is the derivative of the segment's temperature with respect to time. Finally, therefore, we can write

$$\frac{\partial T}{\partial t} = \frac{k_t}{\rho c} \frac{\partial^2 T}{\partial x^2} = K \frac{\partial^2 T}{\partial x^2}, \quad (5)$$

with partial derivatives to denote the idea that  $T$  is now considered a function of both  $x$  and  $t$ . To check the trial solution

$$T(x, t) = T_0 + \frac{A}{\sqrt{t}} e^{-x^2/4Kt} \quad (6)$$

(where  $A$  now represents an arbitrary constant), we just evaluate its derivatives:

$$\begin{aligned} \frac{\partial T}{\partial t} &= -\frac{1}{2} A t^{-3/2} e^{-x^2/4Kt} + \frac{A}{\sqrt{t}} \left( \frac{x^2}{4Kt^2} \right) e^{-x^2/4Kt} = \frac{A}{\sqrt{t}} \left( \frac{x^2}{4Kt^2} - \frac{1}{2t} \right) e^{-x^2/4Kt}; \\ \frac{\partial T}{\partial x} &= \frac{A}{\sqrt{t}} \left( -\frac{x}{2Kt} \right) e^{-x^2/4Kt}; \\ \frac{\partial^2 T}{\partial x^2} &= \frac{A}{\sqrt{t}} \left[ -\frac{1}{2Kt} e^{-x^2/4Kt} + \left( -\frac{x}{2Kt} \right)^2 e^{-x^2/4Kt} \right]. \end{aligned} \quad (7)$$

Multiplying the last line through by  $K$  and comparing to the first line, we see that they agree and so this trial solution does indeed work. Plotted as a function of  $x$ , as shown in the Fig. 2 below. It is a Gaussian "bell curve" that gets shorter and broader as  $t$  increases.

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In[1]:= Plot[Evaluate[Table[Exp[-x^2/(4 t)]/Sqrt[t], {t, 1, 4}]],
{x, -10, 10}]
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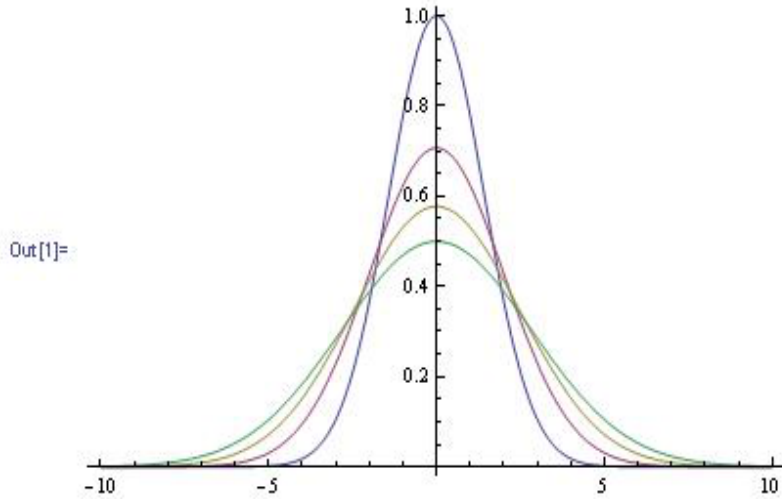


Figure 2: plot for the heat equation solution

Here, I am using units of  $x, t$ , and  $T$  such that  $A$  and  $K$  are both numerically equal to 1. Physically, this solution corresponds to a "pulse" of excess energy that is placed at  $x = 0$  at time  $t = 0$ . At  $t = 0$  the excess energy is all at the same place, so the pulse is infinitely high and narrow. It then spreads out according to Fourier's law, quickly at first and slowly later on.

## 2 Problem 1.68(4+1pts)

Suppose the perfume has already spread over half of the room, densely enough to barely be smelled. If the length of the room is  $\Delta x$  and its cross-sectional area is  $A$ , then we can write, as in equation 1.71,

$$\frac{N}{A\Delta t} \approx D \frac{N/V}{\Delta x} \approx D \frac{N}{A(\Delta x)^2}, \quad (8)$$

with the same  $N$  on each side to indicate that we want a substantial fraction of the molecules to cross from one side of the room to the other. Canceling the  $N$ 's and  $A$ 's and solving for  $\Delta t$  then gives

$$\Delta t \approx \frac{(\Delta x)^2}{D} \approx \frac{(4m)^2}{10^{-5}m^2/s} \approx 10^6 s \approx 2weeks, \quad (9)$$

where I have assumed that the room is a few meters across and used a typical  $D$  value for diffusion through air. Obviously this is much longer than you would actually have to wait to smell the perfume, so *convection* must be the dominant transport mechanism in this situation.

### 3 Problem 1.70(4+1pts)

Consider two "boxed" within the gas, each of width  $l$  (the mean free path) and cross-sectional area  $A$  (see the Fig. 3 below):

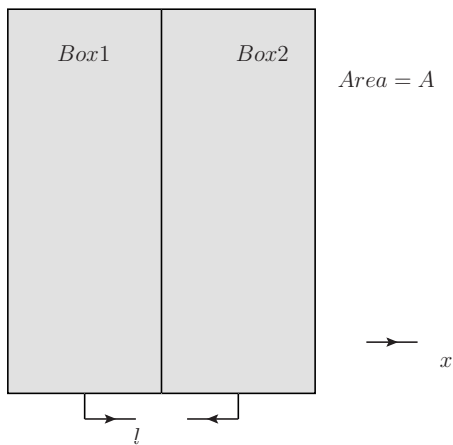


Figure 3: set-up sketch for 1.70

During a time interval  $\Delta t$  equal to the average time between collisions ( $l/\bar{v}$ ), about half the molecules in Box 1 will move to Box 2, while about half the molecules in Box 2 move to Box 1. Denoting the concentrations of molecules within the boxes as  $n_1$  and  $n_2$ , the net migration of molecules from left to right is then roughly

$$\frac{1}{2}n_1Al - \frac{1}{2}n_2Al = \frac{1}{2}Al(n_1 - n_2), \quad (10)$$

and the flux is this quantity divided by  $A$  and by  $\Delta t$ :

$$J_x \approx \frac{Al(n_1 - n_2)}{2A\Delta t} = \frac{l(n_1 - n_2)}{2l/\bar{v}} = -\frac{1}{2}l\bar{v}\frac{dn}{dx}. \quad (11)$$

Comparing to Fick's law (equation 1.70), we can read off the diffusion coefficient as approximately  $\frac{1}{2}l\bar{v}$ . For air at room temperature, this evaluates to

$$D \approx \frac{1}{2}l\bar{v} = \frac{1}{2} (1.5 \times 10^{-7}m) (500m/s) = 4 \times 10^{-5}m^2/s, \quad (12)$$

about twice the experimental value quoted in the text. Since  $l$  is proportional to  $V/N = kT/P$  while  $\bar{v}$  is proportional to  $\sqrt{T}$ , the diffusion coefficient should be proportional to  $T^{3/2}/P$ . Therefore, at fixed pressure, doubling the temperature should increase the diffusion coefficient by a factor of about 2.8.

#### 4 Problem 2.1(4+2+4pts)

(a) Here are all the possible outcomes, listed in Table. 1 , 16 in total: where H is for heads and T is for

TTTT	TTHH	THHH	THTH
HTHH	TTTH	THHT	HHTH
TTHT	HTTH	HHHT	THTT
HTHT	HTTT	HHTT	HHHH

Table 1: Solution for 2.1(a)

tails.

(b) The macrostates are:

$$\begin{aligned}
 0heads, \Omega &= 1, probability = \frac{1}{16} \\
 1heads, \Omega &= 4, probability = \frac{4}{16} \\
 2heads, \Omega &= 6, probability = \frac{6}{16} \\
 3heads, \Omega &= 4, probability = \frac{4}{16} \\
 4heads, \Omega &= 1, probability = \frac{1}{16}
 \end{aligned} \quad (13)$$

(c)

$$\begin{aligned}
N = 4, n = 0, \Omega &= \frac{4!}{0! \cdot 4!} = 1 \\
N = 4, n = 1, \Omega &= \frac{4!}{1! \cdot 3!} = \frac{24}{6} = 4 \\
N = 4, n = 2, \Omega &= \frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6 \\
N = 4, n = 3, \Omega &= \frac{4!}{3! \cdot 1!} = 4 \\
N = 4, n = 4, \Omega &= \frac{4!}{4! \cdot 0!} = 1
\end{aligned}
\tag{14}$$

## 5 Problem 2.3(1+2+2+1+1+1+2pts)

(a) There are two possible states for the first coin, and for each of these, two for the second, and for each of these, two for the third, and so on. So the total number of microstates is  $2^{50} = 1.13 \times 10^{15}$ .

(b) The number of ways of getting exactly 25 heads is,

$$\Omega(25) = \binom{50}{25} = \frac{50!}{(25!)^2} = 1.26 \times 10^{14}.
\tag{15}$$

(c) The probability of getting exactly 25 heads is equal to the fraction of all states that have 25 heads,

$$P(25) = \frac{\Omega(25)}{(2)^{50}} = 0.112
\tag{16}$$

(d) The probability of getting 30 heads would be

$$P(30) = \frac{\Omega(30)}{2^{50}} = \frac{1}{2^{50}} \frac{50!}{(30!)(20!)} = 0.042
\tag{17}$$

(e) The probability of getting 40 heads would be

$$P(40) = \frac{\Omega(40)}{2^{50}} = \frac{1}{2^{50}} \frac{50!}{(40!)(20!)} = 0.0000091
\tag{18}$$

(f) There is only one way to get 50 heads, so the probability would be simply

$$P(50) = \frac{1}{2^{50}} = 8.88 \times 10^{-16}. \quad (19)$$

(g) Here are the results for any number of heads from 0 to 50, listed in Table. 2:

Number of heads	Multiplicity	Probability
0	1	$8.88 \times 10^{-16}$
1	50	$4.44 \times 10^{-14}$
2	1225	$1.09 \times 10^{-12}$
3	19600	$1.74 \times 10^{-11}$
4	1230300	$2.05 \times 10^{-10}$
5	2118760	$1.88 \times 10^{-9}$
6	15890700	$1.41 \times 10^{-8}$
7	99884400	$8.87 \times 10^{-8}$
8	536878650	$4.77 \times 10^{-7}$
9	$2.505 \times 10^9$	$2.23 \times 10^{-6}$
10	$1.027 \times 10^{10}$	$9.12 \times 10^{-6}$
$\vdots$	$\vdots$	$\vdots$

Table 2: Results for 2.3(g)

The plot can be found by using the third column vs. the first in Table. 2 to graph, which is shown in the Fig. 4 below,

## 6 Problem 2.5(2\*6+3pts)

To represent each microstate I use a sequence of digits, for the number of energy units in the first, second, and third oscillators, respectively.

(a)  $N=3, q=4$

400 310 031 220 211  
 040 301 103 202 121  
 004 130 013 022 112

I count 15 microstates, and according to the formula there should be

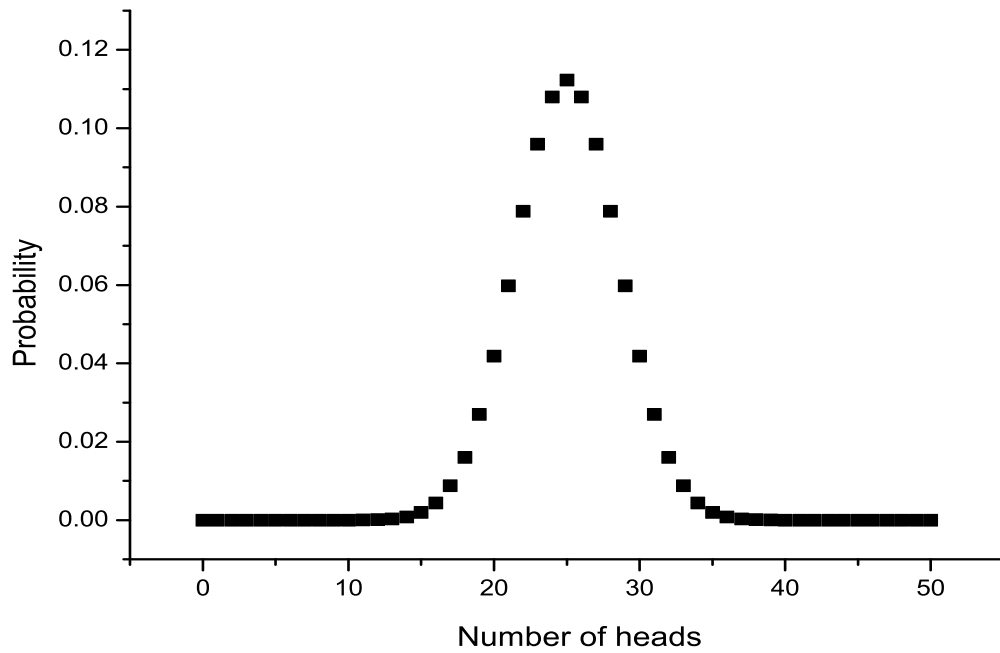


Figure 4: plot for Probability vs. Number of heads

$$\binom{4+3-1}{4} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15.$$

(b)  $N=3, q=5$

500	410	041	320	032	311	221
050	401	104	302	203	131	212
005	140	014	230	023	113	122

I count 21 microstates, and according to the formula there should be

$$\binom{5+3-1}{4} = \frac{7!}{5!2!} = \frac{7 \cdot 6}{2} = 21.$$

(c)  $N=3, q=6$

600	501	015	042	141	033	132
060	150	420	204	114	321	213
006	051	402	024	330	312	123
510	105	240	411	303	231	222

I count 28 microstates, and according to the formula there should be

$$\binom{6+3-1}{6} = \frac{8!}{6!2!} = \frac{8 \cdot 7}{2} = 28.$$

(d)  $N=4, q=2$

2000	0020	1100	1001	0101
0200	0002	1010	0110	0011

I count 10 microstates, and according to the formula there should be

$$\binom{2+4-1}{2} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10.$$

(e)  $N=4, q=3$

3000	2100	0210	0021	1110
0300	2010	0201	1002	1101
0030	2001	1020	0102	1011
0003	1200	0120	0012	0111

I count 20 microstates, and according to the formula there should be

$$\binom{3+4-1}{3} = \frac{6!}{3!3!} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} = 20.$$

(f) if  $N = 1$ , then all the energy must belong to the one and only oscillator, so there is only one microstate, which we would denote simple " $q$ ". And according to the formula, the multiplicity should be

$$\binom{q+1-1}{q} = \frac{q!}{q!} = 1.$$

(g) If  $q = 1$ , then there is only one unit of energy to distribute among the  $N$  oscillators, so the allowed states would be 1000..., 0100..., 0010..., and so on up to ...0001. There are  $N$  places to put the unit of energy, so the number of possible microstates is  $N$ . And indeed, according to the formula,

$$\Omega(N, 1) = \binom{1 + N - 1}{1} = \frac{N!}{1!(N-1)!} = N.$$

## 7 Problem 2.19(5pts)

The multiplicity of a two-state paramagnet is

$$\Omega = \binom{N}{N_{\downarrow}} = \frac{N!}{N_{\downarrow}!(N - N_{\downarrow})!}$$

Applying Stirling's approximation to each of the factorials gives

$$\Omega \approx \frac{(N/e)^N}{(N_{\downarrow}/e)^{N_{\downarrow}} \cdot ((N - N_{\downarrow})/e)^{N - N_{\downarrow}}} = \frac{N^N}{N_{\downarrow}^{N_{\downarrow}} \cdot (N - N_{\downarrow})^{N - N_{\downarrow}}}, \quad (20)$$

up to factors that are merely large. Taking the logarithm of both sides gives

$$\ln \Omega \approx N \ln N - N_{\downarrow} \ln N_{\downarrow} - (N - N_{\downarrow}) \ln (N - N_{\downarrow}). \quad (21)$$

The final logarithm can be written  $\ln [N(1 - N_{\downarrow}/N)] = \ln N + \ln(1 - N_{\downarrow}/N)$ . In the limit  $N_{\downarrow} \ll N$ , the second term is approximately  $-N_{\downarrow}/N$ , so

$$\ln \Omega \approx N \ln N - N_{\downarrow} \ln N_{\downarrow} - (N - N_{\downarrow}) \left( \ln N - \frac{N_{\downarrow}}{N} \right). \quad (22)$$

The  $N \ln N$  terms cancel, while the term  $N_{\downarrow}^2/N$  is negligible compared to the others. Gathering the remaining terms gives

$$\ln \Omega \approx N_{\downarrow} \ln \frac{N}{N_{\downarrow}} + N_{\downarrow} = N_{\downarrow} \ln \frac{N_e}{N_{\downarrow}} = \ln \left( \frac{N_e}{N_{\downarrow}} \right)^{N_{\downarrow}}, \quad (23)$$

so in this limit the multiplicity is approximately

$$\Omega \approx \left( \frac{N_e}{N_\downarrow} \right)^{N_\downarrow}. \quad (24)$$

This is the same result as in problem 2.17, but with  $q$  replaced by  $N_\downarrow$ . The difference between the two systems is that each particle in the paramagnet can hold at most one unit of energy (obtained by flipping the dipole from up to down), while each oscillator in the Einstein solid can hold arbitrarily many units of energy. Apparently, in the limit where the total number of energy units is much less than the number of particles, the states of the Einstein solid in which some oscillators contain more than one unit of energy (each) are insignificant; they contribute a factor to the multiplicity that is merely large.