# PHY306: Homework#3 Solutions, Spring 10

by TA: Yi Gu

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## 1 Problem 2.8(2*5pts)

(a)Of the 20 units of energy, anywhere from 0 to 20 could be in solid $A$. Each possibility from 0 to 20 defines a different macrostate, so there are 21 macrostates in total.

(b)The combined system has 20 oscillators and 20 units of energy, so the total number of microstates is

$$\Omega(20, 20) = \begin{pmatrix} 20 + 20 - 1 \\ 20 \end{pmatrix} = \frac{39!}{20!19!} = 6.89 \times 10^{10}. \tag{1}$$

(c)For the macrostate with all the energy in solid $A$, the multiplicity of solid $A$ is

$$\Omega(10, 20) = \begin{pmatrix} 20 + 10 - 1 \\ 20 \end{pmatrix} = \frac{29!}{20!9!} = 1.00 \times 10^7.$$

while the multiplicity of solid $B$ is 1. Assuming that the system is in equilibrium, all microstates are equally probable, so the probability of this macrostate is

$$Probability = \frac{\Omega(\text{this state})}{\Omega(total)} = \frac{1.00 \times 10^7}{6.89 \times 10^{10}} = 1.45 \times 10^{-4}. \tag{2}$$

(d)For the macrostate with half the energy in each solid, the multiplicyt of the combined system is

$$\Omega = \Omega_A \Omega_B = \begin{pmatrix} 10 + 10 - 1 \\ 10 \end{pmatrix} \begin{pmatrix} 10 + 10 - 1 \\ 10 \end{pmatrix} = \left( \frac{19!}{10!9!} \right)^2 = 8.534 \times 10^9. \tag{3}$$

so the probability (in equilibrium) is

$$Probability = \frac{8.53 \times 10^9}{6.89 \times 10^{10}} = 0.124 \tag{4}$$

(e) The probability of the energy being evenly distributed is greater than that of all the energy being in $A$ by a factor of nearly 1000. So if this system started out with all(or nearly all) of the energy in one solid or the other, then we could be pretty sure that it would evolve toward a state with energy more evenly distributed. And if it started out with the energy evenly distributed, we could be sure that at some later time, we would not find all the energy on one side or the other-this would happen less than one time in a thousand. So the evolution from the unlikely state to the likely one is irreversible, but not exactly since the process does occasionally happen in reverse.

# 2 Problem 2.18(10pts)

First note that $(N-1)! = N!/N$, since dividing by $N$ cancels the final factor in $N!$, leaving just the first$N-1$ factors.Similary, $(q+N-1)! = (q+N)!/(q+N)$. Thus as the hint says,

$$\Omega(N,q)=\begin{pmatrix} q+N-1 \\ q \end{pmatrix} = \tfrac{(q+N-1)!}{q!(N-1)!} = \tfrac{(q+N)!}{q!N!} \cdot \tfrac{N}{q+N}$$

Now applying Sterlling's approximation to each of the factorials and cancel as many factors as possible:

$$\Omega(N,q) \approx \frac{(q+N)^{q+N} e^{-(q+N)} \sqrt{2\pi(q+N)}}{q^q e^{-q} \sqrt{2\pi q} N^N e^{-N} \sqrt{2\pi N}} \cdot \frac{N}{q+N} = \frac{(q+N)^{q+N}}{q^q N^N} \sqrt{\frac{N}{2\pi q(q+N)}}. \tag{5}$$

Finally, write$(q+N)^{q+N}$ as $(q+N)^q \cdot (q+N)^N$, to obtain

$$\Omega(N,q) \approx \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{q}\right)^N \sqrt{\frac{N}{2\pi q(q+N)}}. \tag{6}$$

# 3 Problem 2.28(5pts)

There are 52 possible cards that could be on top, and for each of these choices, there are 51 possibilities for the next card, then 50 for the next, and so on down to 1 choice for the bottom card. So the total number of arrangements is just $52! = 8.06 \times 10^{67}$. If all arrangements are accessible, then the entropy is

$$\frac{S}{k} = ln52! = 156; S = 156k = 2.16 \times 10^{-21} J/K. \tag{7}$$

This is then the amount of entropy created by shuffling the cards, and it's tiny compared to the entropy associated with thermal motions, which is typically a large number(proportional to the number of particles) in fundamental units and a number of order 1 when multiplied by Boltzmann's constant.

# 4 Problem 2.30(5+5+2+3pts)

(a)According to Problem 2.18, the multiplicity of any large Einstein solid is

$$\Omega(N, q) \approx \left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{q}\right)^N \sqrt{\frac{N}{2\pi q(q+N)}}.$$

Here we want to substitute $2N$ for both $N$ and $q$, so the multiplicity reduces to

$$\Omega_{total} = \left(\frac{2N+2N}{2N}\right)^{2N} \left(\frac{2N+2N}{2N}\right)^{2N} \sqrt{\frac{2N}{2\pi 2N(2N+2N)}} = 2^{2N}2^{2N}\sqrt{\frac{1}{8\pi N}} = \frac{2^{4N}}{\sqrt{8\pi N}}. \tag{8}$$

When allmicrostates are allowed, the entropy is

$$\frac{S}{k} = ln\frac{2^{4N}}{\sqrt{8\pi N}} = 4Nln2 - ln\sqrt{8\pi N} = 2.77 \times 10^{23} - 28.1 \tag{9}$$

for $N = 10^{23}$.

(b) The multiplicity of Solid A(only), when it has exactly half of the energy ($q = N$), is

$$\Omega = \left(\frac{N+N}{N}\right)^N \left(\frac{N+N}{N}\right)^N \sqrt{\frac{N}{2\pi N(N+N)}} = 2^N 2^N \sqrt{\frac{1}{4\pi N}} = \frac{2^{2N}}{\sqrt{4\pi N}}. \tag{10}$$

Since the multiplicity of Solid B is the same, the multiplicity of composite system in this macrostate is just the square of $\Omega_A$,

$$\Omega_{\text{most likely}} = \frac{2^{4N}}{4\pi N} \tag{11}$$

Therefore, for the most likely macrostate,

$$\frac{S}{k} = ln\frac{2^{4N}}{4\pi N} = 4Nln2 - ln(4\pi N) = 2.77 \times 10^{23} - 55.5 \tag{12}$$

(c)Notice that the $4Nln2$ terms are the same in both cases above, so the difference between these two entropies is only 55.5-28.1=27.4 units, utterly negligible compared to either of the values themselves. Therefore, the issue of time scales is quite irrelevant for such a large system.

(d)Inserting the partition causes the entropy to decrease by about 27 units out of $2.8 \times 10^{23}$, or about one part in $10^{22}$. I would call this "violation" of the second law *insignificant*.

# 5 Problem 3.1(5+5pts)

In each case I'll use a centered-difference approximation, taking a difference of values just above the just below the point where I want the derivative.When $q_A = 1$

$$T_A = \frac{\Delta U_A}{\Delta S_A} = \frac{2\epsilon - 0\epsilon}{10.7k - 0k} = 0.19\frac{\epsilon}{k} = 220K \tag{13}$$

where the last value is for $\epsilon = 0.1eV$ (so that $\epsilon/k$=$(0.1eV)/(8.62 \times 10^{-5}eV/$K)=1160K). Similarly,

$$T_B = \frac{\Delta U_B}{\Delta S_B} = \frac{100\epsilon - 98\epsilon}{187.5k - 185.3k} = 0.91\frac{\epsilon}{k} = 1060K \tag{14}$$

As expected, Solid B is much hotter when it has nearly all of the energy. However, at $q_A = 60$,

$$T_A = \frac{61\epsilon - 59\epsilon}{160.9k - 157.4k} = 0.57\frac{\epsilon}{k} = 660K \tag{15}$$

while,

$$T_B = \frac{41\epsilon - 39\epsilon}{107.0k - 103.5k} = 0.57\frac{\epsilon}{k} = 660K \tag{16}$$

At this point, the temperatures are essentially the same.

# 6 Problem 3.5

Please refer to solution to homework#4.