PHY306: Homework#4 Solutions, Spring 10

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1 Problem 2.33(10pts)

For argon at room temperature and atmospheric pressure, the volume per molecule is:

$$\frac{V}{N} = \frac{kT}{P} = \frac{(1.38 \times 10^{-23} J/K)(300K)}{10^5 N/m^2} = 4.14 \times 10^{-26} m^3, \tag{1}$$

while the energy per molecule is

$$\frac{U}{N} = \frac{3}{2}kT = \frac{3}{2}(1.38 \times 10^{-23} J/K)(300K) = 6.21 \times 10^{-21} J. \tag{2}$$

The mass of an argon atom is 40 u or 6.64×10^{-26}kg, so the argument of the logarithm in the Sackur-Tetrode equation is

$$\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2} = (4.14 \times 10^{-26} m^3)\left(\frac{4\pi(6.64 \times 10^{-26} kg)(6.21 \times 10^{-21} J)}{3(6.63 \times 10^{-34} Js)^2}\right)^{3/2} = 1.02 \times 10^7. \tag{3}$$

The entropy of a mole of argon under these conditions is therefore

$$S = R[ln(1.02 \times 10^7) + \frac{5}{2}] = R[18.64] = 155 J/K. \tag{4}$$

The only relevant difference between argon and helium in this calculation is the larger mass of the argon atom, which increases the argument of the logarithm by a factor of $(40/4)^{3/2} = 31.6$. The reason why m matters is for a given energy, a molecule with more mass has more momentum, resulting in a larger "hypersphere" of allowed momentum states for the gas and hence a larger multiplicity.

2 Problem 2.35(10pts)

Writing 5/2 as $lne^{5/2}$, the Sackur-Tetrode equation becomes

$$S = Nkln\left[\frac{V}{N}e^{5/2}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right]. \tag{5}$$

We want to know when this quantity is negative, that is ,when the argument of the logarithm is less than 1. So set it equal to 1 and use the equipartition theorem to write U in terms of T:

$$1 = \frac{V}{N}e^{5/2}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2} = \frac{V}{N}e^{5/2}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}. \tag{6}$$

Solving for T gives

$$T = \left(\frac{N}{V}\right)^{2/3}\frac{h^2}{2\pi e^{5/3}mk} \tag{7}$$

We assume that N/V is the same as at room temperature T_0 and atmospheric pressure P_0,sowe can use the ideal gas law to write it as P_0/kT_0, then plug in $P_0 = 10^5$ Pa and $T_0 = 300$ K. The mass of a helium atom is 4 u, where $1u$=1.66×10^{-27} kg. Plugging in all these numbers, I get $T \approx 0.01$ K. Below this temperature, the methods of Chapter 7 become necessary.

3 Problem 3.5(5pts)

The result of Problem 2.17 was

$$\Omega = \left(\frac{eN}{q}\right)^q, \tag{8}$$

for an Einstein solid in the "low-temperature" limit $q \ll N$. Therefore the entropy in this limit is

$$S = kln\left(\frac{eN}{q}\right)^q = kqln\left(\frac{eN}{q}\right) = kq[lne + lnN - lnq] = kq[lnN - lnq + 1]. \tag{9}$$

But $U = q\epsilon$, where ϵ is the size of each energy unit, so

$$S = \frac{kU}{\epsilon}[lnN - lnU + ln\epsilon + 1]. \tag{10}$$

Differentiating with respect to U now gives

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{k}{\epsilon}[ln(N\epsilon/U) + 1] + \frac{kU}{\epsilon}\left(-\frac{1}{U}\right) = \frac{k}{\epsilon}ln\left(\frac{N\epsilon}{U}\right). \tag{11}$$

Solving for U is now just a couple of steps:

$$\frac{\epsilon}{kT} = ln\left(\frac{N\epsilon}{U}\right) \Rightarrow e^{\epsilon/kT} = \frac{N\epsilon}{U} \Rightarrow U = N\epsilon e^{-\epsilon/kT}. \tag{12}$$

Note that as $T \to 0$, the energy goes to zero as expected.

4 Problem 3.10(2+3+2+3pts)

(a) As the ice melts in water, its entropy increases by

$$\triangle s = \frac{Q}{T} = \frac{mL}{T} = \frac{(30g)(333J/g)}{273K} = 36.6J/K. \tag{13}$$

(b) As the water's temperature rises, its entropy increases by

$$\triangle S = \int_{T_i}^{T_f} \frac{CdT}{T} = Cln\frac{T_f}{T_i} = (30g)(4.186J/g \cdot K)ln\frac{298K}{273K} = 11.0J/K. \tag{14}$$

(c) The heat lost by the kitchen is the same as the heat gained by the ice/water, $mL + mc\triangle T$. So the change in the kitchen's entropy is

$$\triangle S = \frac{Q}{T} = \frac{-(30g)(333J/g) - (30g)(4.186J/g \cdot K)(25K)}{298K} = -44.1J/K. \tag{15}$$

(d) The net change in the entropy of the universe due to these events is

$$\triangle S_{total} = 36.6J/K + 11.0J/K - 44.1J/K = 3.5J/K. \tag{16}$$

Since this is an irreversible process, the entropy of the universe has increased (but only slightly, since the temperatures of the ice and the kitchen differed by less than 0.1).

5 Problem 3.16(3+2pts)

(a) Before the memory was erased, it would have been in any one of $2^{2^{33}}$ different microstates (at least). After it is erased, its new microstate is completely specified and unrelated to the previous one, but somehow the whole system, including the hardware that did the erasing, must still have $2^{2^{33}}$ possible states, corresponding to the number of possible initial conditions. this multiplicity of possible states gives the system an entropy equal to

$$S = kln2^{2^{33}} = k \cdot 2^{33}ln2 = k \cdot (6.0 \times 10^9) = 8.2 \times 10^{-14}J/K. \tag{17}$$

(b) To dump this entropy into an environment at 300K would require a heat transfer of at least

$$Q = T\triangle S = (300K)(8.2 \times 10^{-14}J/K) = 2.5 \times 10^{-11}J, \tag{18}$$

or 25 picojoules. Not a very significant amount.

6 Problem 3.20(3+3+3+1pts)

For the numbers given, the quantity $\frac{uB}{kT}$ is

$$x = \frac{\mu B}{kT} = \frac{(9.27 \times 10^{-24} J/T)(2.06T)}{(1.38 \times 10^{-23} J/K)(2.2K)} = 0.629. \tag{19}$$

The hyperbolic tangent of this number is 0.558, so

$$\frac{U}{N\mu B} = -tanhx = -0.558; \frac{M}{N\mu} = tanhx = 0.558 \tag{20}$$

To find the entropy, you could use the equation 3.25 that total energy determines the fractions of up and down dipoles:

$$\frac{N_\uparrow}{N} = \frac{1}{2}\left(1 - \frac{U}{N\mu B}\right) = 0.779; \frac{N_\downarrow}{N} = 1 - \frac{N_\uparrow}{N} = 0.221. \tag{21}$$

From equation 3.28, the maximum possible entropy is $Nkln2$, and the ratio of the actual entropy to the maximum is

$$\frac{S}{S_{max}} = \frac{1}{ln2}\left(lnN - \frac{N_\uparrow}{N}lnN_\uparrow - \frac{N_\downarrow}{N}lnN_\downarrow\right) = -\frac{1}{ln2}\left(\frac{N_\uparrow}{N}ln\frac{N\uparrow}{N} + \frac{N_\downarrow}{N}ln\frac{N_\downarrow}{N}\right) \tag{22}$$

Plugging our numbers into this formula gives 0.76, meaning that the entropy is about 3/4 what it would be if half the dipoles point up and half pointed down. To achieve 99/100 of the maximum magnetization, we would need tanhx=0.99 or x=2.65, about 4.2 times greater than the value for our parameters. So we would need to increase the magnetic field to 4.2 × 2.06T=8.65 T, or decrease the temperature to 2.2K/4.2=0.52K, or combine a somewhat smaller increase in the field strength with a somewhat smaller decrease in the temperature.