APPENDIX

4 Table of Integrals

Here is a table of commonly used integrals involving the Boltzmann, Gaussian, and Planck distributions. a and b are constants.

$$\int_0^\infty e^{-bx}\,dx = \frac{1}{b}$$

$$\int_0^\infty x^2 e^{-bx}\,dx = \frac{2!}{b^3}$$

$$\int_0^\infty x^3 e^{-bx}\,dx = \frac{3!}{b^4}$$

$$\int_0^\infty x^n e^{-bx}\,dx = \frac{n!}{b^{n+1}} \qquad \text{Positive integer n, b > 0}$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi} \qquad \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a} \qquad \int_0^\infty x^2 e^{-ax^2}\,dx = \frac{\sqrt{\pi}}{4a^{3/2}}$$

To find $\int E^{1/2} e^{-bE}\,dE$, etc., use $E = x^2$

$$\int_0^\infty x^3 e^{-ax^2}\,dx = \frac{1}{2a^2} \qquad \int_0^\infty x^4 e^{-ax^2}\,dx = \frac{3\sqrt{\pi}}{8a^{5/2}}$$

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$$

Problem 1

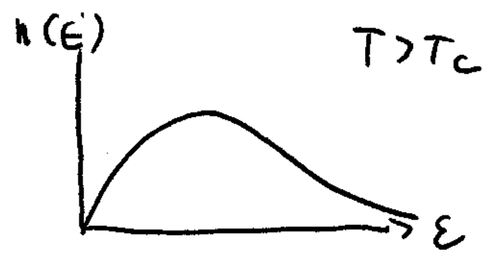
Consider a non-interacting gas of hydrogen atoms (bosons) with the density of $1 \cdot 10^{20} m^{-3}$.

(a) Find the temperature of Bose-Einstein condensation, TC, for this system.

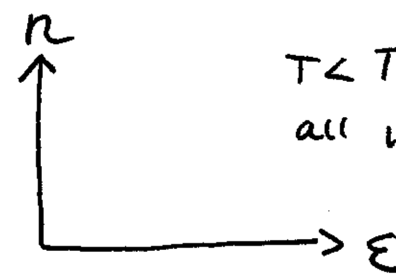
$$T_c = \frac{0.53}{k_B}\left(\frac{h^2}{2\pi m}\right)\left(\frac{N}{V}\right)^{2/3} = \frac{(0.53)(6.64\times10^{-34}\ Js)^2}{(1.38\times10^{-23}\ J/K)(2\pi)(1.7\times10^{-27}\ kg)} (10^{20})^{2/3}$$

$$\sim 3.3\times10^{-5}\ K$$

(b)Draw a qualitative graph of the number of atoms as a function of energy of the atoms for the cases: **T >> TC** and **T = 0.5 TC**.



$T > T_c$ (normal Bose stats)



$T < T_c$
all in $\varepsilon = 0$ state

(c)If the total number of atoms is $1 \cdot 10^{20}$, how many atoms occupy the ground state at **T = 0.5 TC**?

$$N_0 = N - N_E = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] = 10^{20}\left[1 - \left(\frac{1}{2}\right)^{3/2}\right]$$

$$= 0.65\times10^{20}$$

Problem 2

The ground level of the neutral lithium atom is doubly degenerate. The first excited level is 6-fold degenerate and is at an energy 1.2 eV above the ground level.

(a) In the outer atmosphere of the Sun, which is at a temperature of about 6000 K, what fraction of the neutral lithium is in the excited level? Since all the other levels of *Li* are at a much higher energy, it is safe to assume that they are not significantly occupied.

$$Z_1 = \sum_i d_i \exp(-\beta E_i) = 2\exp(-0\beta) + 6\exp(-\beta \cdot 1.2\,\mathrm{eV})$$

$$= 2 + 6\exp(-\beta E_1)$$

$$P(E_1) = \frac{6}{Z} e^{-\beta E_1} = \frac{6e^{-\beta E_1}}{2+6e^{-\beta E_1}} = \frac{3}{e^{\beta E}+3}$$

$$\exp(\beta E) = \exp\left[\frac{1.2\,\mathrm{eV}}{(8.6\times10^{-5}\,\mathrm{eV/K})(6000\,\mathrm{K})}\right] = \exp(2.3) = 10.1$$

$$P(E_1) = 3/(10.1+3) = 0.23 \quad \boxed{23\%}$$

(b) (5) Find the average energy of Li atom at temperature ***T***(again, consider only the ground state and the first excited level).

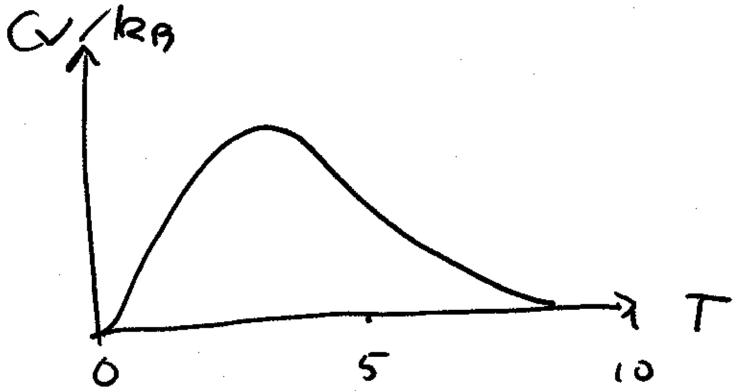
$$\langle E\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{6E\exp(-\beta E)}{2+6\exp(-\beta E)} = \frac{3E}{3\exp(\beta E)+1}$$

$$= \frac{3(1.2\,\mathrm{eV})}{3(10.1)+1} = .096\,(1.2\,\mathrm{eV})$$

(c) Find the contribution of these levels to the specific heat per mole, *CV*, and sketch *CV*as a function of *T*.

$$C_V = \frac{\partial}{\partial T} \langle E \rangle \Big|_V = \frac{-3E \cdot E \exp(\beta E) \frac{\partial \beta}{\partial T}}{[\exp(\beta E) + 3]^2}$$

$$= k_B \frac{3(\beta E)^2 \exp(\beta E)}{\exp(\beta E) + 3}$$



Problem 3

(a) For a system of particles at room temperature, how large must ε-μ be for the Fermi-Dirac, Boltzmann, and Bose-Einstein distributions agree within 1%?

$$\frac{n_{BE}}{n_{FD}} = \frac{\exp\left(\frac{\epsilon-\mu}{k_B T}\right)+1}{\exp\left(\frac{\epsilon-\mu}{k_B T}\right)-1} = 1.01$$

solve for $\underline{\underline{\exp\left(\frac{\epsilon-\mu}{k_B T}\right) \sim 200}}$

so $\frac{\epsilon-\mu}{k_B T} \sim \ln 200 \sim 5.3$ [must be true for all ε]

$\Rightarrow \mu > 5.3\, k_B T$

(b) Estimate the density of a system of mobile electrons in a semiconductor that can be treated at room temperature equally well (with 1% accuracy) using all three distributions. Assume that the effective electron mass is the same as a free electron mass, and that you can use for this estimate the expression for μ in an ideal classical gas.

Here we will use $\mu \sim \mu_{Boltz} = k_B T \ln\left(\frac{n_Q}{n}\right)$

$\ln \frac{n_Q}{n} > 5.3$ so $\frac{n_Q}{n} > e^{5.3} = \underline{\underline{200}}$

$$n_Q = \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} = \left[\frac{(6.28)(9.1\times10^{-31}\text{ kg})(1.38\times10^{-23}\text{ J/K})(300\text{K})}{(6.65\times10^{-34}\text{ J s})^2}\right]^{3/2}$$

$= 1.25\times10^{25}/m^3$

$n < 6.3\times10^{22}/m^3$